Some remarks on the generalized Noether theory of point symmetry transformations of the Lagrangian

1992 J. Phys. A: Math. Gen. 25 991

(http://iopscience.iop.org/0305-4470/25/4/033)

J. Phys. A: Math. Gen. 25 (1992) 991–1007. Printed in the UK

Some remarks on the generalized Noether theory of point symmetry transformations of the Lagrangian

J Krause

Facultad de Física, Pontificia Universidad Católica de Chile, Casilla 6177, Santiago 22, Chile

Received 11 October 1991

Abstract. The Noether theory of infinitesimal point symmetry transformations is revisited and generalized under the broad scope of the general theory of transformations of Lagrangian mechanics. The basic generalized point symmetries of a Lagrangian function are thus obtained and briefly discussed. It is proved that (under very general and physically reasonable provisions) the point symmetry group of the Lagrangian is necessarily a finite Lie group, and a rather simple technique is then introduced for the explicit calculation of the associated Lie algebra. Next, the algebra obeyed by the set of basic Noether quantities (which also includes the traditional Noether constants of motion) is examined. In this way it is shown that not all the generalized point symmetries of a Lagrangian yield an associated conservation law. This paper presents three simple examples of this (not so well known) aspect of the Noether theory.

1. Introduction

In two previous papers (Aguirre and Krause 1991a, b), a brief review of the general theory of point transformations in Lagrangian mechanics has been presented, as an introduction to theory of symmetry and conservation laws. We devote this paper to examining a generalization of the traditional Noether (1918) theory of point transformations under this general conceptual framework.

The relationship between symmetry and conservation laws plays a major role in physics. This relationship arises from an intimate connection between geometry and dynamics, which was firmly established by the pioneering work of Sophus Lie, Felix Klein, Emmy Noether and others (Kastrup 1983). It also appears in the Einstein theory of general relativity as one of its most beautiful crowning achievements. The general principle relating symmetry groups and conservation law was first determined by Emmy Noether, who stated it in almost complete generality. In fact, Noether's theorems afford the oldest (and the first mathematically rigorous) instance of this relationship. Although, during recent years, an interesting endeavour has been accomplished by many authors, in the sense of extending this relation beyond the traditional realm of the Noether theory (Currie and Saletan 1966, Lutzky 1979a, b, 1981, Hojman and Harleston 1981, Prince and Eliezer 1981, Prince 1983a, b). Our intention in the present series of papers is to give a systematic review of this important subject of contemporary classical mechanics. However, in these papers we also present some new interesting results concerning this issue (cf also Aguirre *et al* 1991).

0305-4470/92/040991+17$04.50

It is unnecessary to emphasize the tremendous impact that the Noether theorems have had on theoretical physics over the years. Let us recall here only that, in her famous paper, Emmy Noether proves two theorems which form the basis of all group-theoretic approaches to conservation laws in physics (and that also prompted those non-Noether approaches introduced in recent literature). (Cf, for instance, Konopleva and Popov 1981 for a good discussion of both theorems within the context of classical field theory.) The first theorem is referred to the case of symmetry of a Lagrangian theory with respect to a r-dimensional (i.e. finite) Lie group of local point transformations. Her second theorem is valid in the case of symmetry with respect to an 'infinite continuous group', i.e. a pseudo-group (whose transformations are not diffeomorphisms of configuration spacetime, to be sure, for they are functions of the derivatives of the dependent variables and, moreover, they also depend on r arbitrary functions and their derivatives, instead of r arbitrary parameters) (Konopleva and Popov 1981). In this paper we deal only with the first theorem.

It is conventional in the literature on Noether's theorem to reserve the name Noether symmetries for generators of variational symmetries of the action functional; namely, *for infinitesimal 'active' point transformations* which keep the form of the Lagrangian function invariant (see e.g. Cantrijn and Sarlet 1981). This convention is useful in distinguishing these symmetries from other important group actions in the theory (Prince 1982, 1983, 1985). We shall generalize this notion. The generalized version of the theorem presented here requires some knowledge of the point symmetry theory of Lagrangian mechanics, as developed in our previous papers. This version is perhaps not familar to most physicists. Though we keep in it the main features of Noether's first theorem, our approach contains some novelties. Here we shall concentrate our attention only on Lagrangian systems with a finite number of degrees of freedom (although our novel results can be extended to Lagrangian field theory where they may play an important role). Besides this restriction, we would like to remark on the following aspects of our approach:

(i) We do not interpret the infinitesimal diffeomorphisms as variational symmetries of the action functional, as one usually does in this subject.

(ii) Rather, we interpret them as infinitesimal coordinate transformations (in configuration spacetime), which keep invariant the form of the Lagrangian function by the addition of a suitable infinitesimal gauge transformation.

(iii) We also take into account the eventual change of scale induced by the required symmetry transformations, i.e. we use 'gauge-scaling constants' in the present formalism

(iv) We do not assume a closed Lie algebra, obeyed by the generators of the symmetry transformation. Rather, the resulting maximal finite Lie algebra of the infinitesimal point symmetries of a given Lagrangian appears as a necessary consequence of the formalism, and we teach the reader how to *calculate the realization* of this algebra in an explicit way.

(v) Next, an intimate relation between the scaling constants and the structure constants of the algebra is found. (By the way, the existence of this relation neatly shows that the scaling constants are far from being a spurious trivial element of the Lagrangian symmetry formalism.)

(vi) Finally, in this approach we obtain a simplified version of the classical algebra obeyed by the Noether quantities (which are the analogues of the Noether currents of classical field theory, for systems with a finite number of degrees of freedom), and we show that not all the generalized point symmetries of a Lagrangian function yield an associated conservation law.

These comments fix the context of this paper, which also includes three miscellaneous examples of the generalized formalism of Noether's point symmetry theory.

One final remark is perhaps not out of place in this introduction. We are aware of the tools and techniques from differential geometry that have been developed specifically, over the last 30 years, for studying the calculus of variations, and in particular the theory for Lagrangian systems arising from a Lagrangian (see e.g. Cantrijn and Sarlet 1981, Prince 1982, 1983, 1985). However, we are also aware of the fact that the use and the understanding of these powerful (and beautiful) techniques are reserved for the specialist, since they do not belong to the current mathematical curriculum of most physicists (who, on the other hand, are all interested in Noether's theorem). Although in this paper we have avoided the use of differential forms, there are several occasions in which the language of differential forms would be rather effective. In fact, the main tools used in this article are the prolongations of Lie's vector fields. As is well known, differential forms are well suited to the study of the geometric theory of the calculus of variations (Olver 1986). Note that, in this paper, point symmetries arise as coordinate transformations in configuration spacetime, rather than as variational transformations. Hence, it would appear, at first sight, that the results in this paper are not formulated in an intrinsic (global) fashion for motion of dynamical systems in arbitrary manifolds. We would deem such a remark as more formal than real, because it is well known since the times of Kretschmann (1917) and Einstein (1918) that every Lagrangian theory is automatically generally covariant. This geometric (i.e. absolute) fact is not a mere consequence of the notation, or of the mathematical technique, one uses to handle the formalism.

2. Infinitesimal coordinate transformations in configuration spacetime

Only finite local transformations were considered in our previous papers. We now briefly discuss the change induced in the description of a Lagrangian system by an infinitesimal transformation of coordinates in configuration spacetime. Let the transformation be given by

$$T = t + \varepsilon u(t, q) \qquad Q^j = q^j + \varepsilon v^j(t, q) \tag{2.1}$$

where ε is a parameter of smallness, $0 < \varepsilon \ll 1$. (Henceforth all expansions will be taken to the first order of approximation in ε, as usual.) The generalized velocities $\mathring{Q}^j$, associated with the new coordinates, are given by

$$\mathring{Q}^j = \dot{q}^j + \varepsilon(\dot{v}^j - \dot{q}^j \dot{u}) \tag{2.2}$$

where, clearly, $\dot{u} = \mathrm{d}u/\mathrm{d}t = u_t + u_j \dot{q}^j$, with $u_t = \partial u/\partial t$, $u_j = \partial u/\partial q^j$, and so forth. Thus, for instance, for any given function $F(t, q, \dot{q})$, one has the first-order expansion

$$F(T, Q, \mathring{Q}) = F(t, q, \dot{q}) + \varepsilon\left(u\frac{\partial F}{\partial t} + v^j\frac{\partial F}{\partial q^j}\right) + \varepsilon(\dot{v}^j - \dot{q}^j\dot{u})\frac{\partial F}{\partial \dot{q}^j}. \tag{2.3}$$

It is useful to introduce the generator $v^{[0]}$ of the infinitesimal transformation of coordinates (2.1), which is given by the vector field

$$v^{[0]} \equiv u(t, q)\partial/\partial t + v^j(t, q)\partial/\partial q^j \tag{2.4}$$

as well as the first prolongation of this differential operator, which is defined as

$$v^{[1]} \equiv v^{[0]} + (\dot{v}^j - \dot{q}^j\dot{u})\partial/\partial\dot{q}^j \tag{2.5}$$

(Olver 1986). (Higher prolongations will not be needed in the present work.) In this fashion, equation (2.3) can be written briefly as

$$F(T, Q, \mathring{Q}) = (1 + \varepsilon v^{[1]}) F(t, q, \dot{q}). \tag{2.6}$$

It is clear that by means of equations (2.1) one can obtain a new Lagrangian system that differs by 'very little' from the old system represented by $L(t, q, \dot{q})$. This new system can be characterized by a Lagrangian function $\hat{L}(T, Q, \mathring{Q})$ given by (cf Aguirre and Krause 1991a)

$$\mathrm{d}T\,\hat{L}(T, Q, \mathring{Q}) = \mathrm{d}t(1 + \varepsilon K) L(t, q, \dot{q}) + \varepsilon\, \mathrm{d}G(t, q) \tag{2.7}$$

where the constant K and the gauge function $G(t, q)$ are arbitrary. (Of course, one takes $1 + \varepsilon K$ and $\varepsilon G(t, q)$, in equation (2.7), in order to have $\hat{L} = L + 0(\varepsilon)$.)

The motivation for equation (2.7) follows. We face here a transformation (i.e. equation (2.1)) which we can interpret either from a 'passive' or from an 'active' viewpoint. Although it usually matters little which intuitive point we adopt, at this stage we get a better development of these topics by presenting them under the scope of the 'passive' point of view (which is also more akin to the theory of relativity). Thus, equation (2.1) will be thought of as a local transformation of coordinates in configuration spacetime. We next consider the action integral S from this point of view. In order to calculate a value for the functional S, one has to specify a curve $q^j = c^j(t)$; one then evaluates the action integral along the chosen curve, with $\dot{q}^j = \mathrm{d}c^j(t)/\mathrm{d}t$. In this fashion, given a transformation of coordinates, one writes $q^j = q^j(T, Q) = c^j[t(T, Q)]$, from which the expression $Q^j = C^j(T)$ for the curve follows in terms of the new coordinates (provided the conditions required by the implicit function theorem are satisfied). Hence we write, quite generally,

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L(t, q, \dot{q}) = \int_{T_1}^{T_2} \mathrm{d}T\, \hat{L}(T, Q, \mathring{Q}) = \hat{S} \tag{2.8}$$

where we define the new Lagrangian by

$$\dot{T}\hat{L}(T, Q, \mathring{Q}) = L(t, q, \dot{q}). \tag{2.9}$$

On the right-hand side of equation (2.8) we integrate along $Q^j = C^j(T)$ between the limits $T_1 = T[t_1, c(t_1)]$ and $T_2 = T[t_2, c(t_2)]$, since T is the new variable of integration. Note that equation (2.8) is valid for every chosen curve $q^j = c^j(t)$ whatsoever. In other words, equation (2.8) entails a simple change of variables in an integral, and therefore no question of symmetry for S is involved here. Moreover, according to equation (2.9), one proves that gauge transformations of the Lagrangian are invariant under general coordinate transformations in configuration spacetime. In this fashion, one justifies the following definition: every local transformation of coordinates in configuration spacetime induces a new class of Lagrangian functions $\hat{L}$, which can be defined by

$$\dot{T}\hat{L}(T, Q, \mathring{Q}) = KL(t, q, \dot{q}) + \dot{G}(t, q) \tag{2.10}$$

where L is the old Lagrangian, G an arbitrary gauge function, K an arbitrary gauge constant and T is the new independent variable. This definition makes sense, because equation (2.10) differs from equation (2.9) by an arbitrary gauge transformation. (Certainly, instead of equation (2.8), one now has $\hat{S} = KS + G_2 - G_1$, which corresponds to a gauge transformation of the action functional (see Levy-Leblond 1979).) In the sequel we shall refer to equation (2.10) as a Lagrangian transformation induced by a local coordinate transformation in configuration spacetime (cf also Camprin and Prince 1985 for some interesting comments on equivalent Lagrangians).

Clearly, equation (2.7) corresponds to an infinitesimal Lagrangian transformation. A straightforward expansion in equation (2.7) then yields

$$\hat{L}(t, q, \dot{q}) = L(t, q, \dot{q}) + \varepsilon(KL + \dot{G} - \dot{u}L - v^{[1]}L). \tag{2.11}$$

This equation entails the general change of form of the Lagrangian that is induced by an infinitesimal coordinate transformation in configuration spacetime, to within an arbitrary infinitesimal gauge transformation.

The next task is to obtain an equivalent expression for the change of form of the Lagrangian stated in equation (2.11), which is also very useful. To this end, one tries to introduce in the right-hand side of equation (2.11) as many total time derivatives as possible. After some manipulations, one obtains the following result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{u\left(\dot{q}^j\frac{\partial L}{\partial \dot{q}^j} - L\right) - v^j\frac{\partial L}{\partial \dot{q}^j}\right\} = -\dot{u}L - v^{[1]}L + (v^j - \dot{q}^j u)\frac{\delta L}{\delta q^j}. \tag{2.12}$$

From the standpoint of mechanics, this is certainly an interesting (and well known) result, because, besides the appearance of the variational derivatives

$$\frac{\delta L}{\delta q^j} = \frac{\partial L}{\partial q^j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^j}\right) \tag{2.13}$$

one also recognizes in equation (2.12) the presence of the generalized energy function (cf Desloge 1982)

$$E(t, q, \dot{q}) = \dot{q}^j\frac{\partial L}{\partial \dot{q}^j} - L \tag{2.14}$$

as well as the generalized momenta

$$p_j(t, q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^j}. \tag{2.15}$$

Hence, equation (2.11) can be written in the following equivalent form:

$$\hat{L}(t, q, \dot{q}) = L(t, q, \dot{q}) + \varepsilon\left(\frac{\mathrm{d}}{\mathrm{d}t}(uE - v^j p_j + G) + KL - (v_j - \dot{q}^j u)\frac{\delta L}{\delta q^j}\right). \tag{2.16}$$

This formula corresponds to the 'infinitesimal version' of the general finite law of transformation of a Lagrangian already discussed by Aguirre and Krause (1991a), and plays a central role in the Noether theory of point symmetries.

3. The Noether criterion formula for the infinitesimal point symmetries of the Lagrangian

We now begin the study of the Noether theory of infinitesimal point symmetry transformations, adopting for this purpose the 'passive' standpoint for interpreting diffeomorphisms in configuration spacetime. In this interpretation one refers the point symmetries of a system directly to the chosen representative Lagrangian function, instead of considering them as infinitesimal variational symmetries of the action functional. The theory obtained in this fashion coincides in its essential features with the theory provided by the standard variational approach; however, it also contains some new interesting results.

Let us first enounce the following result: the necessary and sufficient condition for an infinitesimal coordinate transformation

$$\hat{t} = t + \varepsilon u(t, q) \qquad \hat{q}^j = q^j + \varepsilon v^j(t, q) \tag{3.1}$$

to be a point symmetry of the Lagrangian $L(t, q, \dot{q})$ is that there exist a constant κ and a function $\sigma(t, q)$ such that

$$\mathrm{d}\hat{t}\, L(\hat{t}, \hat{q}, \mathring{\hat{q}}) = \mathrm{d}t(1 + \varepsilon\kappa)L(t, q, \dot{q}) + \varepsilon\, \mathrm{d}\sigma(t, q) \tag{3.2}$$

holds, where $\mathring{\hat{q}}^j = \dot{q}^j + \varepsilon(\dot{v}^j - \dot{q}^j\dot{u})$. The proof of this fact is rather simple, if one requires that $L(t, q, \dot{q})$ is not a 'null' Lagrangian, i.e. $L(t, q, \dot{q}) \neq \dot{f}(t, q)$. Note that the Lagrangian function L that figures in the RHS of equation (3.2) has the same form as that in the LHS. Although it looks rather obvious, this is an important result, for it tells us that the expression (3.2) for an infinitesimal point symmetry of L is unique, i.e. there is no remaining gauge freedom associated with an infinitesimal point symmetry transformation of L.

In this way, according to equation (2.11), one also has the following result: a necessary and sufficient condition for $u(t, q)$ and $v^j(t, q)$ to be generators of a point symmetry of $L(t, q, \dot{q})$ is that there exist a function $\sigma(t, q)$ and a constant κ, such that

$$\dot{\sigma} = \dot{u}L + v^{[1]}L - \kappa L \tag{3.3}$$

holds for all values of $(t, q, \dot{q})$ where (3.2) is well defined. This formula corresponds to the Noether criterion for the generalized infinitesimal point symmetries of a given Lagrangian function.

Once L is given, equation (3.3) provides a test which must be satisfied by the functions u, v^j and σ, and by the constant κ, in order to qualify as generators of a point symmetry of L. Furthermore, this formula provides a linear homogeneous equation for the determination of u, v^j, σ and κ. Assume that $\{u_1, v_1^j, \sigma_1, \kappa_1\}$ and $\{u_2, v_2^j, \sigma_2, \kappa_2\}$ are two solutions to equation (3.3); then any linear combination of these certainly provides another solution of (3.3). Though equation (3.3) is clearly not an eigenvalue equation, the problem it sets is very similar to an eigenvalue problem, since one has to solve this problem simultaneously for those admissible functions $\{u_a, v_a^j, \sigma_a\}$ which are associated with the admissible constant κ_a. Namely, equation (3.3) corresponds to an 'algebraic-differential' linear homogeneous problem (as eigenvalue differential equations also do). For future reference, let us write equation (3.3) more explicitly. It reads

$$\frac{\partial L}{\partial \dot{q}^j}\left\{\left(\frac{\partial v^j}{\partial t} + \dot{q}^k \frac{\partial v^j}{\partial q^k}\right) - \dot{q}^j\left(\frac{\partial u}{\partial t} + \dot{q}^k \frac{\partial u}{\partial q^k}\right)\right\} + L\left(\frac{\partial u}{\partial t} + \dot{q}^j \frac{\partial u}{\partial q^j}\right)$$
$$- \left(\frac{\partial \sigma}{\partial t} + \dot{q}^j \frac{\partial \sigma}{\partial q^j}\right) + \frac{\partial L}{\partial t} u + \frac{\partial L}{\partial q^j} vj - L\kappa = 0. \tag{3.4}$$

By means of equation (2.11), one can also write the Noether criterion formula (3.3) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(uE - v^j p_j + \sigma) = (v^j - \dot{q}^j)\frac{\delta L}{\delta q^j} - \kappa L \tag{3.5}$$

which holds for all $(t, q, \dot{q})$ and is independent of the $\ddot{q}$. In fact, using equation (2.12), after some manipulations, equation (3.5) can be cast in the equivalent form

$$E\dot{u} - p_j\dot{v}^j + \dot{\sigma} = v^{[0]}L - \kappa L \tag{3.6}$$

where all terms containing the $\ddot{q}$ manifestly cancel out. Equations (3.3)-(3.6) are completely equivalent, for they correspond to different ways of writing the Noether criterion formula. They all appear in the current literature (with $\kappa = 0$); as a matter of fact, they have different important uses.

4. The basic point symmetries of the Lagrangian

The Noether criterion formula has a constructive character, to which we now turn our attention. Equation (3.3) affords a system of equations which must be satisfied by the generators, and the associated gauge transformation leading to an infinitesimal Lagrangian transformation that keeps L invariant. One obtains these equations in the following manner.

Given the Lagrangian L as a known function of t, q and $\dot{q}$, one expands equation (3.3) in terms of the power of the generalized velocities, observing that the $\dot{q}$ do not appear as arguments in the unknown functions $u(t, q)$, $v^j(t, q)$ and $\sigma(t, q)$. In consequence, by means of the expansion coefficients of the different powers of the $\dot{q}$ (once suitably symmetrized), one separates equation (3.3) into a system of linear homogeneous equations obeyed by $\{u, v^j, \sigma, \kappa\}$.

For instance, for the sake of concreteness, let us assume that the Lagrangian is of the standard form

$$L(t, q, \dot{q}) = \tfrac{1}{2}\Lambda_{jk}(t, q)\dot{q}^j\dot{q}^k + \Gamma_j(t, q)\dot{q}^j - \Phi(t, q) \tag{4.1}$$

with $\Lambda_{jk} = \Lambda_{kj}$. In this case, equation (3.4) becomes separated into the following system:

$$\Lambda_{jk}\frac{\partial u}{\partial q^m} + \Lambda_{mj}\frac{\partial u}{\partial q^k} + \Lambda_{km}\frac{\partial u}{\partial q^j} = 0 \tag{4.2a}$$

$$-\Lambda_{jk}\frac{\partial u}{\partial t} + \Lambda_{jm}\frac{\partial v^m}{\partial q^k} + \Lambda_{km}\frac{\partial v^m}{\partial q^j} + \frac{\partial \Lambda_{jk}}{\partial t}u + \frac{\partial \Lambda_{jk}}{\partial q^m}v^m + \Lambda_{jk}\kappa = 0 \tag{4.2b}$$

$$\Phi\frac{\partial u}{\partial q^j} - \Gamma_k\frac{\partial v^k}{\partial q^j} + \frac{\partial \sigma}{\partial q^j} - \frac{\partial \Gamma_j}{\partial t}u - \Lambda_{jk}\frac{\partial v^k}{\partial t} - \frac{\partial \Gamma_j}{\partial q^k}jv^k + \Gamma_j\kappa = 0 \tag{4.2c}$$

$$\Phi\frac{\partial u}{\partial t} + \frac{\partial \sigma}{\partial t} + \frac{\partial \Phi}{\partial t}u + \frac{\partial \Phi}{\partial q^j}v^j - \Gamma_j\frac{\partial v^j}{\partial t} + \Phi\kappa = 0 \tag{4.2d}$$

which is linear and homogeneous indeed. In the most interesting case of a non-singular Lagrangian system (i.e. when $\Lambda(t, q) = \det(\Lambda_{jk}) \neq 0$), equation (4.2$a$) yields $\partial u/\partial q^j = 0$, for $j = 1, \ldots, n$, and these equations simplify a great deal.

It must be remarked that this approach is very general, since it can be applied to all polynomial Lagrangian functions of the $\dot{q}$. Moreover, it can be applied under rather weak assumptions concerning the analytic properties of the Lagrangian as a function of the $\dot{q}$. (Plainly, a negative power of some generalized velocity $\dot{q}^j$ is ruled out in every reasonable Lagrangian function one can conceive, for otherwise the mechanical system would not be well defined at $\dot{q}^j = 0$.) It is also important to remark that the constant κ (i.e. the scaling factor $1 + \varepsilon\kappa$) is one of the unknowns of the problem.

Thus, we see that under very general analytical provisions on the function L (which are also physically reasonable) one does always get a linear homogeneous system for the determination of u, v^j, σ and κ (as in equations (4.2) for instance). Hence, the principle of superposition holds for the problem posed by equation (3.3), and the general solution reads

$$u(t, q) = k^1 u_1 + \ldots + k^r u_r \equiv k^a u_a(t, q) \tag{4.3a}$$

$$v^j(t, q) = k^1 v_1^j + \ldots + k^r v_r^j \equiv k^a v_a^j(t, q) \tag{4.3b}$$

$$\sigma(t, q) = q^1 \sigma_1 + \ldots + k^r \sigma_r \equiv k^a \sigma_a(t, q) \tag{4.3c}$$

$$\kappa = k^1 \kappa_1 + \ldots + k^r \kappa_r \equiv k^a \kappa_a \tag{4.3d}$$

where $\{u_a, v_a^j, \sigma_a, \kappa_a; a = 1, \ldots; r\}$ is a set of basic solutions to equation (3.3), and the k are arbitrary constants. So one has (cf equation (3.3))

$$\dot{\sigma}_a = \dot{u}_a L + v_a^{[1]} L - \kappa_a L \tag{4.4}$$

or, for that matter (cf equation (3.6)),

$$E\dot{u}_a - p_j \dot{v}_a^j + \dot{\sigma}_a = v_a^{[0]} L - \kappa_a L \tag{4.5}$$

for $a = 1, \ldots, r$.

Usually, most of the κ_a turn out to be zero. However, there are many Lagrangians for which some of the κ_a are non-zeroth. It is clear that basic solutions of this kind correspond to valid symmetries of L, and there is no *a priori* reason to disregard them. In fact, if one sets $\kappa = 0$ in equation (3.3) from the beginning, one is artificially reducing the number of point symmetries of L, since in this way one obtains only a subset of basic solutions to equation (3.3), in a rather arbitrary fashion. On the other hand, there also exist many examples of Lagrangians for which equation (3.3) has no solution at all, besides the trivial one; namely: $u = 0$, $v^j = 0$, $\sigma = 0$ and $\kappa = 0$.

Of course, the mathematical structure of the linear homogeneous system of equations obeyed by $\{u, v^j, \sigma, \kappa\}$, and hence the nature of the general solution (in particular the dimension r of its basic solution space) depends exclusively on the functional form of the Lagrangian. We shall further illuminate this matter by means of some examples (see section 7).

Fortunately, it is not necessary to have complete information about the mathematical structure of the problem set by equation (3.3), in each particular case, in order to continue with the general analysis of this subject. The fact that this is a linear homogeneous problem (under all mechanically reasonable circumstances) is enough for this purpose.

5. The Lie algebra $N(L)$ of the group $G(L)$

We next discuss one of the main features of the Noether theory, which is seldom considered in detail in the current literature. The group $G(L)$ of all point symmetries of a Lagrangian L was introduced by Aguirre and Krause (1991b), and its general structure was examined by means of the finite elements. It will now be shown that the infinitesimal elements of $G(L)$ constitute a closed finite Lie algebra, and therefore it follows that $G(L)$ is always a finite Lie group.

The following lemmas shall be needed presently. We omit the proofs here; lemma 1 can be proved in a direct way, and lemmas 2 and 3 are well known (Olver 1986).

Lemma 1. For any function $\varphi(t, q)$, one has

$$\frac{d}{dt}(v^{[0]}\varphi) = \dot{u}\dot{\varphi} + v^{[1]}\dot{\varphi}. \tag{5.1}$$

Lemma 2. Let $v_a^{[0]}$ and $v_b^{[0]}$ be two generators of infinitesimal point transformations, i.e.

$$v_a^{[0]} = u_a(t, q)\partial/\partial t + v_a^j(t, q)\partial/\partial q^j \qquad v_b^{[0]} = u_b(t, q)\partial/\partial t + v_b^j(t, q)\partial/\partial q^j \tag{5.2}$$

then the commutator

$$[v_a^{[0]}, v_b^{[0]}] \equiv v_{ab}^{[0]} = u_{ab}\partial/\partial t + v_{ab}^j\partial/\partial q^j \tag{5.3}$$

is given by

$$u_{ab}(t, q) = v_a^{[0]}u_b - v_b^{[0]}u_a \qquad v_{ab}^j(t, q) = v_a^{[0]}v_b^j - v_b^{[0]}v_a^j. \tag{5.4}$$

Moreover, for the commutator of their first prolongations one also has

$$[v_a^{[1]}, v_b^{[1]}] \equiv v_{ab}^{[1]} = v_{ab}^{[0]} + (\dot{v}_{ab}^j - \dot{q}^j\dot{u}_{ab})\partial/\partial\dot{q}^j. \tag{5.5}$$

Lemma 3. If a set of point generators satisfy the Lie algebra

$$[v_a^{[0]}, v_b^{[0]}] = f_{ab}^c v_c^{[0]} \tag{5.6}$$

where f_{ab}^a denotes the structure constants, then all their higher prolongations satisfy the same Lie algebra. In particular, one has

$$[v_a^{[1]}, v_b^{[1]}] = f_{ab}^c v_c^{[1]} \tag{5.7}$$

for the first prolongations. Furthermore, in such cases the commutators (5.6) (cf also equations (5.3) and (5.4)) are given by

$$u_{ab}(t, q) = f_{ab}^c u_c(t, q) \qquad v_{ab}^j(t, q) = f_{ab}^c v_c^j(t, q). \tag{5.8}$$

We are now ready to prove the following theorem.

Theorem 1. The basic point symmetry generators $\{u_a, v_a^j; a = 1, \ldots, r\}$ of a Lagrangian function consitute a basis for the realization of a r-dimensional Lie algebra.

Proof. Let us define the following functions:

$$\sigma_{ab}(t, q) \equiv (v_a^{[0]}\sigma_b - u_b^{[0]}\sigma_a) - (\kappa_a\sigma_b - \kappa_b\sigma_a) \tag{5.9}$$

where clearly each (κ_a, σ_a) corresponds to that gauge transformation which is associated with the point symmetry of L generated by (u_a, v_a^j). We then evaluate $\dot{\sigma}_{ab}$ in a straightforward manner, using for this purpose equation (4.4) and lemmas 1 and 2. Thus, we get

$$\begin{aligned}
&\dot{\sigma}_{ab} + (\kappa_a\dot{\sigma}_b - \kappa_b\dot{\sigma}_a) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}(v_a^{[0]}\sigma_b - v_b^{[0]}\sigma_a) \\
&\quad = \dot{u}_a\dot{\sigma}_b + v_a^{[1]}\dot{\sigma}_b - \dot{u}_b\dot{\sigma}_a - v_b^{[1]}\dot{\sigma}_a \\
&\quad = \dot{u}_a(\dot{\sigma}_b - \dot{u}_b L) - \dot{u}_b(\dot{\sigma}_a - \dot{u}_a L) + v_a^{[1]}(\dot{u}_b L + v_b^{[1]}L - \kappa_b L) \\
&\qquad - v_b^{[1]}(\dot{u}_a L + v_a^{[1]} - \kappa_a L) \\
&\quad = \dot{u}_a(v_b^{[1]}L - \kappa_b L) - \dot{u}_b(v_a^{[1]}L - \kappa_a L) + v_a^{[1]}(\dot{u}_b L) - v_b^{[1]}(\dot{u}_a L) \\
&\qquad + [v_a^{[1]}, v_b^{[1]}]L - \kappa_b v_a^{[1]}L + \kappa_a v_b^{[1]}L \\
&\quad = \kappa_a(\dot{u}_b L + v_b^{[1]}L) - \kappa_b(\dot{u}_a L + v_a^{[1]}L) + L(v_a^{[1]}\dot{u}_b - v_b^{[1]}\dot{u}_a) + v_{ab}^{[1]}L \\
&\quad = \kappa_a(\dot{\sigma}_b + \kappa_b L) - \kappa_b(\dot{\sigma}_a + \kappa_a L) + v_{ab}^{[1]}L + L\frac{\mathrm{d}}{\mathrm{d}t}(v_a^{[0]}u_b) - \dot{u}_a\dot{u}_b \\
&\qquad - \frac{\mathrm{d}}{\mathrm{d}t}(v_b^{[0]}u_a) + \dot{u}_b\dot{u}_a \\
&\quad = \kappa_a\dot{\sigma}_b - \kappa_b\dot{\sigma}_a + v_{ab}^{[1]}L + \dot{u}_{ab}L
\end{aligned} \tag{5.10}$$

i.e.

$$\dot{\sigma}_{ab} = \dot{u}_{ab} L + v^{[1]}_{ab} L. \tag{5.11}$$

This result evidently corresponds to equation (4.4), with

$$\kappa_{ab} = 0. \tag{5.12}$$

This means that the generators (u_{ab}, v^j_{ab}) of the commutator $v^{[0]}_{ab}$ (as defined in equations (5.4)) and the associated gauge function σ_{ab} (defined in equation (5.9)) satisfy the Noether formula (3.3), with the particular value $\kappa_{ab} = 0$ for the constant κ. Hence, as a consequence of the superposition principle (cf equations (4.3)), it follows that equation (5.6) holds. This finishes the proof of the theorem. □

Let us denote by $N(L)$ the Lie algebra obeyed by a basic set of point symmetry generators of L (we shall briefly refer to this algebra as the Noether algebra of L). The result stated in equation (5.12) is particularly interesting. In order to clarify its meaning, we shall prove the following theorem.

Theorem 2. Let $\{u_a, v^j_a;\ 1 \leqslant a \leqslant r\}$ be a set of basic point symmetry generators, which satisfy the Noether algebra $N(L)$, and let $\{\kappa_a, \sigma_a;\ 1 \leqslant a \leqslant r\}$ be the set of associated symmetry gauge transformations, then one has

$$\sigma_{ab}(t, q) = f^c_{ab} \sigma_c(t, q) \tag{5.13}$$

and

$$f^c_{ab} \kappa_c = 0 \tag{5.14}$$

where σ_{ab} is defined in equation (5.9) and f^c_{ab} denotes the structure constants of $N(L)$.

Proof. Let us define the functions $\hat{\sigma}_{ab} = f^c_{ab} \sigma_c$; we have

$$\begin{aligned} \dot{\hat{\sigma}}_{ab} &= f^c_{ab} \dot{\sigma}_c = f^c_{ab} (\dot{u}_c L + v^{[1]}_c L - \kappa_c L) \\ &= \dot{u}_{ab} L + v^{[1]}_{ab} L - f^c_{ab} \kappa_c L \\ &= \dot{\sigma}_{ab} - f^c_{ab} \kappa_c L \end{aligned} \tag{5.15}$$

where we have used equations (5.8) and (5.11). Now, we see that according to equation (5.15), equation (5.13) $\Rightarrow$ equation (5.14). Furthermore, if one has $\dot{\hat{\sigma}}_{ab} \neq \dot{\sigma}_{ab}$ and $f^c_{ab} \kappa_c \neq 0$, for some $a \neq b$, one would have a 'null Lagrangian', i.e.

$$L = \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \frac{\sigma_{ab}(t, q) - \hat{\sigma}_{ab}(t, q)}{f^c_{ab} \kappa_c} \right\} \Rightarrow \frac{\delta L}{\delta q^j} \equiv 0 \tag{5.16}$$

which is certainly not the case. (The addition of a constant to the σ is immaterial, of course.) Hence, equations (5.13) and (5.14) follow, which proves the theorem. □

Note that it is not possible to produce a proof that $N(L)$ is a finite Lie algebra, by means of purely differential form manipulations or otherwise, without recourse to solving the Noether symmetry criterion formula for obtaining $u_a(t; q)$, $v^j_a(t; q)$, $\sigma_a(t, q)$ and κ_a, with $a = 1, \ldots, r < \infty$, and using the superposition principle. Furthermore, it is well known that for systems with a finite number of degrees of freedom (as we are discussing here) the set of all generators of $N(L)$ forms a Lie algebra which must be finite-dimensional because it is a subalgebra of the finite algebra of generators of point symmetries of the equations of motion (Sarlet 1983). This fact, however, does not lessen the interest of the analysis presented in this paper because (i) it stems from the non-Noether realm of the formalism, and (ii) the Noether theory has its own virtues that make it interesting in its own right. For instance, this indirect argument for proving the finiteness of $N(L)$ breaks down for systems with infinite degrees of freedom (i.e. continuous fields), because the algebra of generators of point symmetries of the field equations of motion is, in general, infinite (Olver 1986); while, on the other hand, the Noether approach (as presented in this paper) can still give a finite $N(L)$ algebra in the case of a Lagrangian field theory. (This matter will be discussed in a forthcoming paper.) Interesting as it is, the non-Noether formalism has not superseded the Noether theory.

6. The Noether quantities and constants of motion

We devote this section to presenting a very brief review of the famous corollary of Noether's first theorem, which establishes the intimate relationship between point symmetries and conservation laws in Lagrangian mechanics. Equation (3.5) is particularly interesting in this sense, for it immediately yields the following result.

Theorem 3. If $u(t, q)$, $v^j(t, q)$ and $\sigma(t, q)$ correspond to a point symmetry of $L(t, q, \dot{q})$, such that $\kappa = 0$, then

$$J(t, q, \dot{q}) = uE - v^j p_j + \sigma \tag{6.1}$$

is a constant of motion, i.e.

$$\dot{J} \doteq 0. \tag{6.2}$$

The proof is immediate, because on the physical trajectories of the system one has $\delta L/\delta q^j \doteq 0$, for $j = 1, \ldots, n$. This is one of the most interesting theorems of mechanics. However, let us remark that in the present approach it is not true that every point symmetry of a Lagrangian yields a conservation law. In fact, if u, v^j and σ are solutions of equation (3.3) such that $\kappa \neq 0$, then one gets

$$\dot{J} \doteq -\kappa L \tag{6.3}$$

instead of equation (6.2). In other words, a point symmetry which is committed with a change of scale in the associated gauge transformation of the Lagrangian does not give rise to a Noether constant of motion. In this case, if one integrates equation (6.3) along a physical trajectory of the system, one gets $J_2 - J_1 \doteq -\kappa S$, where S is the value of the action integral over the chosen physical path, instead of $J_2 - J_1 \doteq 0$ as one obtains when $\kappa = 0$. Notwithstanding this feature, let us note a few facts here:

(i) One has a symmetry of the Lagrangian, in a strict sense, even when it turns out that $1 + \varepsilon\kappa \neq 1$ (cf equation (3.2)).

(ii) One gets $\kappa \neq 0$ not only as a trivial consequence of scaling the configuration spacetime coordinates.

(iii) If one sets $\kappa = 0$ from the beginning, one is contriving to obtain only a subalgebra $N_0(L)$ of the full Noether algebra $N(L)$ of the Lagrangian.

According to the results obtained in section 4, with each basic point symmetry generator $v_a^{[0]}$ of L one can associate a quantity J_a given by

$$J_a(t, q, \dot{q}) = u_a E - v_a^j p_j + \sigma_a \tag{6.4}$$

$a=1,\ldots,r$. These quantities are linearly independent and, certainly, on the physical trajectories of the system, they satisfy

$$\dot{J}_a \doteq -\kappa_a L \tag{6.5}$$

in general. These quantities are the analogues of the 'Noether currents' of classical field theory. We shall call them the fundamental Noether quantities of the mechanical system. Note that only some of these Noether quantities correspond to the Noether constants of motion; namely, those with $\kappa_a = 0$.

It is also interesting to consider the Noether quantities J_{ab} associated with the commutator $v_{ab}^{[0]}$ of the algebra $N(L)$, i.e.

$$J_{ab}(t, q, \dot{q}) = u_{ab}E - v_{ab}^j p_j + \sigma_{ab}. \tag{6.6}$$

We can easily analyse these quantities by means of equations (5.8) and (5.13); thus we get $J_{ab} = f_{ab}^c(u_c E - v_c^j p_j + \sigma_c)$, which means

$$J_{ab} = f_{ab}^c J_c. \tag{6.7}$$

The most interesting property of these 'commutator quantities' stems from equations (5.14) and (6.5), since one has $\dot{J}_{ab} = -f_{ab}^c \kappa_c L$; that is

$$\dot{J}_{ab} \doteq 0 \tag{6.8}$$

for all $a \neq b$. Hence, all the Noether quantities associated with the Lie brackets (i.e. with the commutators of the Noether algebra) are constants of motion. From a group-theoretic point of view, this means that the subalgebra $N_0(L)$ (namely, the usual algebra of infinitesimal symmetries of L, with $k=0$) is an ideal of $N(L)$.

One could say that these considerations settle the basic features of a classical 'algebra of currents' for Lagrangian systems with a finite number of degrees of freedom.

7. Three miscellaneous examples

Finally, in this section we present three applications of the generalized Noether theory of point symmetry transformations in Lagrangian mechanics, which serve to illustrate some important points of the formalism.

7.1. 1D free particle

Let us consider the following 1D Lagrangian system:

$$L_0(\dot{q}) = \tfrac{1}{2}\dot{q}^2. \tag{7.1}$$

In order to find the point symmetry generators of L_0, we have to solve equation (3.3), which now reads

$$\dot{q}\left(\left(\frac{\partial v}{\partial t} + \dot{q}\frac{\partial v}{\partial q}\right) - \dot{q}\left(\frac{\partial u}{\partial t} + \dot{q}\frac{\partial u}{\partial q}\right)\right) + \frac{1}{2}\dot{q}^2\left(\frac{\partial u}{\partial t} + \dot{q}\frac{\partial u}{\partial q}\right) - \left(\frac{\partial \sigma}{\partial t} + \dot{q}\frac{\partial \sigma}{\partial q}\right) - \frac{1}{2}\dot{q}^2\kappa = 0. \tag{7.2}$$

Since this equation must be satisfied by all values of $\dot{q}$, it becomes separated into the following system:

$$\frac{\partial u}{\partial q} = 0 \qquad 2\frac{\partial v}{\partial q} - \frac{\partial u}{\partial t} - \kappa = 0 \qquad \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial q} = 0 \qquad \frac{\partial \sigma}{\partial t} = 0. \tag{7.3}$$

One readily integrates these equations. The general solution is given by

$$u = k_1 t + k_2 + k_3 t^2 \qquad v = k_4 q + k_5 t + k_6 + k_3 tq \tag{7.4}$$

and

$$\sigma = \tfrac{1}{2}k_3 q^2 + k_5 q \qquad \kappa = 2k_4 - k_1 \tag{7.5}$$

where the k are constants of integration, which play the role of the infinitesimal parameters of the group $G(L_0)$. The finite realizations of the point symmetry group $G(L_0)$ were found in one of our previous papers (Aguirre and Krause 1991b). The reader can easily check that

$$d\hat{t}\,\mathring{\hat{q}}^2 = [1 + \varepsilon(2k_4 - k_1)]\dot{q}^2 + 2\varepsilon d(\tfrac{1}{2}k_3 q^2 + k_5 q) \tag{7.6}$$

holds (to the first order of approximation in ε) under the following infinitesimal point transformations:

$$\hat{t} = t + \varepsilon(k_1 t + k_2 + k_3 t^2) \qquad \hat{q} = q + \varepsilon(k_4 q + k_5 t + k_6 + k_3 tq). \tag{7.7}$$

Table 1 presents the six basic solutions $\{u_a, v_a, \sigma_a, \kappa_a\}$, the associated Noether currents J_a, and the point symmetry generators $v_a^{[0]}$, with $1 \leqslant a \leqslant 6$, for the Lagrangian $\frac{1}{2}\dot{q}^2$. The Lie algebra of $G(L_0)$ (i.e. the Noether algebra $N(L_0)$) is shown in table 2, from where we read the non-zeroth structure constants, i.e.

$$f^2_{21} = f^3_{13} = f^5_{15} = f^4_{23} = f^6_{25} = f^5_{63} = f^5_{54} = f^6_{64} = 1 \qquad \text{and} \qquad f^1_{23} = 2.$$

Note that the general rule stated in equation (5.14) is satisfied: $f^1_{23}\kappa_1 + f^4_{23}\kappa_4 = 0$, since from equations (7.5) we see that the only non-zeroth κ'_a are $\kappa_1 = -1$ and $\kappa_4 = 2$. As for the Noether 'currents' associated with the basic commutators of $N(L_0)$, the only one

Table 1. The basic point symmetry generators $\{u_a, v_a\}$, the corresponding symmetry gauge transformations $\{\sigma_a, \kappa_a\}$, the associated Noether quantities J_a, and the point symmetry generators $v_a^{[0]}$, of the free particle standard Lagrangian $L_0 = \frac{1}{2}\dot{q}^2$.

a	μ_a	v_a	σ_a	κ_a	J_a	$\dot{J}_a$	$v_a^{[0]}$
1	t	0	0	-1	$\frac{1}{2}t\dot{q}^2$	$-\kappa_1 L_0$	$t(\partial/\partial t)$
2	1	0	0	0	$\frac{1}{2}\dot{q}^2$	0	$\partial/\partial t$
3	t^2	tq	$\frac{1}{2}q^2$	0	$\frac{1}{2}(q - t\dot{q})^2$	0	$t^2(\partial/\partial t) + tq(\partial/\partial q)$
4	0	q	0	2	$-q\dot{q}$	$-\kappa_4 L_0$	$q(\partial/\partial q)$
5	0	t	q	0	$q - t\dot{q}$	0	$t(\partial/\partial q)$
6	0	1	0	0	$-\dot{q}$	0	$\partial/\partial q$

Table 2. The Noether algebra $N(L_0)$ of the Lagrangian function $L_0 = \frac{1}{2}\dot{q}^2$. One obtains the commutator $[v_a^{[0]}, v_b^{[0]}]$ at the intersection of the ath row with the bth column.

	$v_1^{[0]}$	$v_2^{[0]}$	$v_3^{[0]}$	$v_4^{[0]}$	$v_5^{[0]}$	$v_6^{[0]}$
$v_1^{[0]}$	0	$-v_2^{[0]}$	$v_3^{[0]}$	0	$v_5^{[0]}$	0
$v_2^{[0]}$	$v_2^{[0]}$	0	$2v_1^{[0]} + v_4^{[0]}$	0	$v_6^{[0]}$	0
$v_3^{[0]}$	$-v_3^{[0]}$	$-2v_1^{[0]} - v_4^{[0]}$	0	0	0	$-v_5^{[0]}$
$v_4^{[0]}$	0	0	0	0	$-v_5^{[0]}$	$-v_6^{[0]}$
$v_5^{[0]}$	$-v_5^{[0]}$	$-v_6^{[0]}$	0	$v_5^{[0]}$	0	0
$v_6^{[0]}$	0	0	$v_5^{[0]}$	$v_6^{[0]}$	0	0

of interest is $J_{23}=f^1_{23}J_1+f^4_{23}J_4=2J_1+J_2$. Thus, one has $J_{23}=\dot{q}(t\dot{q}-q)$, which certainly yields $\dot{J}_{23}\doteq 0$.

7.2. *The simple harmonic oscillator*

We next discuss the generalized point symmetries of $L=\frac{1}{2}(\dot{q}^2-\omega^2q^2)$. Instead of directly solving equation (3.3) in this case, we take advantage of the fact that $\frac{1}{2}(\dot{q}^2-\omega^2q^2)$ and $\frac{1}{2}\mathring{Q}^2$ are *c*-equivalent Lagrangians under the diffeomorphisms $T=\tan\omega t$, $Q=q\sec\omega t$, with $G(t,q)=\frac{1}{2}q\tan\omega t$ and $K=\omega^{-1}$ (cf Aguirre and Krause 1991a). Hence, let us write

$$\hat{T}=\tan\omega\hat{t}=T+\varepsilon(k_1T+k_2+k_3T^2)=\tan\omega t+\varepsilon(k_1\tan\omega t+k_2+k_3\tan^2\omega t) \tag{7.8a}$$

$$\hat{Q}=\hat{q}\sec\omega\hat{t}=Q+\varepsilon(k_4Q+k_5T+k_6+k_3TQ)$$
$$=q\sec\omega t+\varepsilon(k_4q\sec\omega t+k_5\tan\omega t+k_6+k_3q\sec\omega t\tan\omega t) \tag{7.8b}$$

where, clearly, $\hat{t}=t+\varepsilon u$ and $\hat{q}=q+\varepsilon v$. In this way, one readily obtains the generators u and v:

$$\omega u(t)=k_1\sin\omega t\cos\omega t+k_2\cos^2\omega t+k_3\sin^2\omega t \tag{7.9a}$$

$$v(t,q)=-k_1q\sin^2\omega t-k_2q\sin\omega t\cos\omega t+k_3q\sin\omega t\cos\omega t$$
$$+k_4q+k_5\sin\omega t+k_6\cos\omega t. \tag{7.9b}$$

In order to obtain the corresponding symmetry gauge transformation (cf equation (3.1)) using this approach, one has to recall the formula (Aguirre and Krause 1991a)

$$\hat{\sigma}(T,Q)=K\sigma(t,q)+G(\hat{t},\hat{q})-\kappa G(t,q).$$

For an infinitesimal point symmetry transformation, this formula yields

$$\hat{\sigma}(T,Q)=K\sigma(t,q)-\kappa G(t,q)+v^{[0]}G(t,q). \tag{7.10}$$

Hence, since in the present example one has $\hat{\sigma}=\frac{1}{2}k_3q^2+k_5Q$, $G=\frac{1}{2}q^2\tan\omega t$, $K=\omega^{-1}$ and $\kappa=2k_4-k_1$ (cf equations (7.5)), after some simple manipulations, one obtains

$$\sigma(t,q)=-k_1\omega q^2\sin\omega t\cos\omega t-(k_2-k_3)(\omega/2)q^2(\cos^2\omega t-\sin^2\omega t)$$
$$+k_5\omega q\cos\omega t-k_6\omega q\sin\omega t. \tag{7.11}$$

In this fashion one solves equation (3.3) for the standard Lagrangian of the simple harmonic oscillator, and one gets the six basic Noether symmetries of this Lagrangian system. The Noether quantities of this system are shown in table 3, and the Noether algebra is the same as shown in table 2.

Table 3. Gauge-scaling constant κ_a, Noether quantities J_a, and Noether constants of motion $\dot{J}_a\doteq 0$ ($a=2,3,5$ and 6), for the standard Lagrangian of the simple harmonic oscillator.

k_a	κ_a	J_a	$\dot{J}_a=-\kappa_aL$
k_1	-1	$\frac{1}{2\omega}(\dot{q}^2-\omega^2q^2)\sin\omega t\cos\omega t+q\dot{q}\sin^2\omega t$	$\frac{1}{2}(\dot{q}^2-\omega^2q^2)$
k_2	0	$\frac{1}{2\omega}(\dot{q}^2\cos^2\omega t+\omega^2q^2\sin^2\omega t)+q\dot{q}\sin\omega t\cos\omega t$	0
k_3	0	$\frac{1}{2\omega}(\dot{q}^2\sin^2\omega t+\omega^2q^2\cos^2\omega t)-q\dot{q}\sin\omega t\cos\omega t$	0
k_4	2	$-q\dot{q}$	$-(\dot{q}^2-\omega^2q^2)$
k_5	0	$-\dot{q}\sin\omega t+\omega q\cos\omega t$	0
k_6	0	$-\dot{q}\cos\omega t-\omega q\sin\omega t$	0

Of course, the Lagrangian $\frac{1}{2}(\dot{q}^2-\omega^2 q)$ is invariant under time translation: $\hat{t}=t+\hat{k}_2$ (say). One can remedy this omission in the previous formalism by defining a new set of parameters (which corresponds to a change of basis in the algebra $N(L)$). As for setting $\kappa=0$, we wish to remark only that one obtains a 5D subalgebra, and that in this case the Noether J indeed correspond to five constants of motion.

7.3. The 3D Kepler system

Finally, let us briefly consider the following Lagrangian in ordinary space:

$$L(\boldsymbol{x},\dot{\boldsymbol{x}})=\tfrac{1}{2}\dot{\boldsymbol{x}}^2+\frac{K}{r} \tag{7.12}$$

where $K>0$ is a constant, and $r^2=\boldsymbol{x}^2$, with $\boldsymbol{x}=(x,y,z)=(x^1,x^2,x^3)$. In this case, equations (4.2) yield

$$\frac{\partial v^j}{\partial x^k}+\frac{\partial v^k}{\partial x^j}-\delta_{jk}[\dot{u}(t)+\kappa]=0 \tag{7.13a}$$

$$\frac{\partial v^j}{\partial t}-\frac{\partial \sigma}{\partial x^j}=0 \tag{7.13b}$$

$$\frac{K}{r^3}\delta_{jk}x^j v^k-\frac{K}{r}[\dot{u}(t)-\kappa]+\frac{\partial\sigma}{\partial t}=0. \tag{7.13c}$$

One easily integrates these equations; the general solution reads

$$u(t)=3k_5 t+k_4 \qquad v^j(x)=2k_5x^j+\varepsilon_{jkm}k^m x^k \tag{7.14}$$

$$\sigma=0 \qquad \kappa=k_5. \tag{7.15}$$

Hence, in the present formalism, the Noether quantities of a Kepler system correspond to

$$J_5=\tfrac{3}{2}t\dot{\boldsymbol{x}}^2-2\boldsymbol{x}\cdot\dot{\boldsymbol{x}}-\frac{3Kt}{r} \tag{7.16}$$

$$J_4=\tfrac{1}{2}\dot{\boldsymbol{x}}^2-\frac{K}{r} \tag{7.17}$$

and

$$J_j=\varepsilon_{jkm}x^k\dot{x}^m \qquad j=1,2,3. \tag{7.18}$$

The meaning of J_4 and J_j is clear. Thus, one has

$$\dot{J}_j\doteq 0 \qquad \dot{J}_4\doteq 0 \qquad \dot{J}_5\doteq -L \tag{7.19}$$

as required (cf also Prince and Eliezer 1981, Prince 1983).

8. Concluding remarks

In the enormous amount of literature on Noether's theorem, and on symmetries and conservation laws in Lagrangian mechanics, it has always been emphasized that to every point symmetry shown by a given Lagrangian there corresponds an associated

Noether current which is a constant of motion (i.e. the traditional version of Noether's first theorem). In this paper we have proven that this is not always the case. Moreover, this has been done not only by means of some few counter-examples; rather, we obtain this important result within the context of a general theory of point symmetries of the Lagrangian (Aguirre and Krause 1991a, b).

It is indeed astonishing to learn that change-of-scale symmetry transformations (as genuine point symmetries of the Lagrangian) are not related to Noether constants of motion. The fact that they can be regarded as non-Noether constants of motion (namely, as trivial symmetries of the Euler–Lagrange equations) does not preclude the importance of this result. For instance, one does not quantize a system through its equations of motion.

There is no reason to disregard the use of changes of scale in physics. One must be aware of the enormous importance that scale transformations have in classical theoretical mechanics and engineering. On the other hand, changes of scale are of great interest in contemporary Lagrangian quantum theories. (Let us recall the renormalization manipulations, which play such a vital role in them.) One could speculate, for example, that the main result contained in this paper can shed some light on the present problem of anomalies in gauge field quantum theories (cf Jackiw 1985a, b), since the Ward identities are the quantum counterpart of the classically conserved Noether currents (cf Itzykson and Zuber 1980). Of course, there is no room in the space allotted here to go into these quantum considerations. (This matter will be discussed elsewhere.)

The final point we wish to make here is that the Noether theorem of Lagrangian mechanics yields one of the most important formalisms of physics, and the better we know it, the better will be our understanding of many physical theories which are (or can be) embedded in the Lagrangian framework.

Acknowledgments

The author acknowledges several remarks made by the referees and the adjudicator to a previous version of this paper, which have helped to improve it. This work was supported in part by FONDECYT, under Contract 0629/91.

References

Aguirre M, Friedli C and Krause J 1991 SL(3, *R*) as the group of symmetry transformations for all one-dimensional linear systems. III. Equivalent Lagrangian formalisms *Preprint* submitted to *J. Math. Phys.*

Aguirre M and Krause J 1991a *Int. J. Theor. Phys.* **30** 495

—— 1991b *Int. J. Theor. Phys.* **30** 1461

Camprin M and Prince G 1985 *Phys. Lett.* **108A** 191

Cantrijn F and Sarlet W 1981 *SIAM Rev.* **23** 467

Currie D G and Saletan W 1966 *J. Math. Phys.* **7** 967

Desloge E A 1982 *Classical Mechanics* vol 2 (New York: Wiley)

Einstein A 1918 *Ann. d. Phys.* **54** 241

Hill E L 1951 *Rev. Mod. Phys.* **23** 253

Hojman S and Harleston J 1981 *J. Math. Phys.* **22** 1414

Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: McGraw-Hill)

Jackiw R 1985a *Comments Nucl. Part. Phys.* **15** 99

—— 1985b *Recent Developments in Quantum Field Theory* ed J Ambjorn *et al* (Amsterdam: Elsevier) pp 203ff
Kastrup H A 1983 The contributions of Emmy Noether, Felix Klein and Sophus Lie to the modern concept of symmetries in physical systems *Preprint* Institut für Theoretische Physik, RWTH Aachen, 5100 Aachen, Federal Republic of Germany
Konopleva N P and Popov V N 1981 *Gauge Fields* (London: Harwood)
Kretschmann E 1917 *Ann. Phys.* **53** 575
Levy-Leblond J M 1979 *Commun. Math. Phys.* **12** 64
Lutzky M 1979a *Phys. Lett.* **72A** 86
—— 1979b *Phys. Lett.* **75A** 8
—— 1981 *J. Math. Phys.* **22** 1628
Noether E 1918 *Nachr. König. Gessel. Wissen. Göttingen Math. Phys. Kl.* 235–57
Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)
Prince G 1982 *Bull. Austral. Math. Soc.* **25** 309
—— 1983a *Bull. Austral. Math. Soc.* **27** 53
—— 1983b *J. Phys. A: Math. Gen.* **16** L105
—— 1985 *Bull. Austral. Math. Soc.* **32** 299
Prince G E and Eliezer C J 1981 *J. Phys. A: Math. Gen.* **14** 587
Sarlet W 1983 *J. Phys. A: Math. Gen.* **16** L229